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# $\mathcal{N}=4$ superconformal mechanics as a non linear realization

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Abstract: An action for a superconformal particle is constructed using the non linear realization method for the group $PSU(1,1|2)$, without introducing superfields. The connection between $PSU(1,1|2)$ and black hole physics is discussed. The lagrangian contains six arbitrary constants and describes a non-BPS superconformal particle. The BPS case is obtained if a precise relation between the constants in the lagrangian is verified, which implies that the action becomes kappa-symmetric.

Keywords: D-branes, Extended Supersymmetry.

## Contents



## 1. Introduction

The $\mathcal{N} = 4$ superconformal symmetry appears in the dynamics of a charged particle in the near horizon geometry of a four-dimensional charged extremal black hole [1]. The connection can be traced back to the geometry present in this case, which has the structure of $AdS_2 \times S^2$. This implies that the mechanics describing the radial motion of the charged particle in the near horizon geometry inherits the global conformal symmetry group in one dimension, $\mathrm{SO}(1,2)$, [2]. The near horizon geometry of the charged four-dimensional extremal black hole is given by the Bertotti-Robinson metric[1],

$$ds^2 = -\left(\frac{\rho}{M}\right)^2 dt^2 + \left(\frac{M}{\rho}\right)^2 d\rho^2 + M^2 d\Omega^2. \tag{1.1}$$

[1] For a review see [3, 4]

This geometry admits eight globally defined real Killing spinors (it is a BPS state with four real local supersymmetries), which implies the existence of 8 real supercharges. Hence, a simple mechanical model which captures this property is the $PSU(1,1|2)$ conformal mechanics.[2] The superfield equations of motion can be constructed using the method of nonlinear realizations (**NLR**) in superspace [5, 6]. N-particle superconformal mechanics has been studied in [7], where the relation with black hole physics is analyzed. $\mathcal{N}=4$ superconformal mechanics also arises in the computation of the macroscopic black hole entropy of a D0-D4 black hole [8].

In this paper we will study more in general the dynamics of a superconformal particle. This dynamical action is constructed by the method of non-linear realizations without using superfields or requiring additional constraints [9]. As in [5, 6], we consider the coset $PSU(1,1|2)$ and no notion of geometry is used to construct the action. The Goldstone fields will depend only on the world line parameter $\tau$. This procedure allows us to consider in a unified framework the cases of broken and unbroken supersymmetries. The lagrangian will depend on six couplings constants, whose physical meaning is associated with characteristics of the particle and the black hole, like mass, charge, angular momentum. In the case with unbroken supersymmetries, a new local gauge symmetry, kappa symmetry, appears so that half of the fermionic fields can be gauged away and a BPS lagrangian is obtained. This symmetry appears when the coupling constants verify a precise relation. This condition can be understood in two ways, as an equality between the Casimir invariants of the SU(2) and the SO$(1,2)$ sectors, or more physically, as the equality $m=e$, where $m$ and $e$ are the mass and the charge of the particle. In this case, the existence of supercharges, $Q$ and $S$, generating standard supersymmetry and superconformal transformations, respectively, allows to consider two kinds of BPS configurations; those that saturate the bound of the hamiltonian $H=\left[Q,Q^{\dagger}\right]_{+}$, and those saturating the bound of the special conformal transformation generator $K=\left[S,S^{\dagger}\right]_{+}$.

In the superconformal model considered here, there also appear two bosonic local symmetries, one corresponds to ordinary diffeomorphisms of the world line, and the other is a U(1) gauge symmetry. The gauge symmetries appearing in this model are understood as a right action of the coset following reference [10]. The U(1) symmetry only transforms the Goldstone fields associated to SU(2) coordinates. Putting the fermions to zero two decoupled lagrangians are obtained: i) the conformal mechanics lagrangian written in a diffeomorphism invariant form, and ii) the lagrangian of a particle on a sphere in which a monopole is located at the center. The latter system has only one degree of freedom, in agreement with the existence of the U(1) gauge symmetry. If the fermions are switched on, the two systems interact but the U(1) symmetry still remains.

It is well known that, at the quantum level, the conformal mechanics has no ground state associated to the hamiltonian $H$ and the wave function spreads out to spatial infinity. In [2], de Alfaro, Fubini and Furlan suggested that one should consider the eigenstates of the compact operator $P_0=\frac{1}{2}(H+K)$ which has a discrete spectrum and normalizable

[2]This group is often referred to as $SU(1,1|2)$, although this group contains nontrivial central extensions, which are absent in $PSU(1,1|2)$.

eigenstates. From the perspective of the particle motion near the black hole, this corresponds to a different choice of time [1]. In fact, the variable conjugate to $P_0$ is the global time of $\mathrm{AdS}_2$ and can describe the motion of the particle entering in horizon, instead the time conjugate to $H$ only describes the motion of the particle outside the horizon. Therefore, it is also natural to study the dynamics of the superconformal particle using the new basis, that we call the *AdS basis*[3]. In our approach this implies a new parametrization of the coset, leading to a new parametrization for the action (see section 5). The system is now described by a relativistic lagrangian containing two square roots, plus a WZ term that represents the coupling of the particle to the electromagnetic field. This lagrangian has also three gauge invariances as in the previous parametrization, also referred to as the *conformal basis*.

In summary, the $\mathcal{N}=4$ super conformal model, which is presented here in two different basis, describes the equatorial motion of a particle in the background of a near horizon of a $\mathcal{N}=2$, charged, four-dimensional, extremal black hole. A $D(2,1,\alpha)$ superconformal mechanics in superfield formalism of [12] describes also a motion of a particle in a equatorial plane. A general three-dimensional motion in $\mathcal{N}=4$ conformal mechanics [8], [13, 14][4] is not obtained with the coset considered here. It is natural to ask whether there exits other cosets that can produce a general three-dimensional motion without further physical or geometrical requirements.

The outline of the paper is as follows. In section 2, the Maurer-Cartan (**MC**) forms are constructed, and in section 3 the Lagrangian in the conformal basis is presented. In section 4 the gauge symmetries of the model and the gauge fixed form of the lagrangian are studied, and in section 5 an *AdS* parametrization of the coset is given. section 6 is devoted to discussions. There are five appendices with some technical details.

## 2. The $PSU(1,1|2)$ Lie algebra and its NLR

The essential feature of the MC forms that make them useful objects to describe dynamical systems is that they define invariants under a non-linearly realized group action. The first step to calculate them is to choose a coset, in this case, it follows from the discussion in the introduction that the choice will be $PSU(1,1|2)$. The associated algebra is formed by generators of dilatation $D$, special conformal transformations $K$, time translations $H$, SU(2) rotations $J_a$, four supersymmetries $Q^i$, $Q^\dagger_i$, and four superconformal symmetries $S^i$, $S^\dagger_i$. The algebra is given in the appendix A.

Then it is possible to locally parametrize an arbitrary supergroup element $g$ as: [5]

$$g = g_0 e^{i(Q\eta^\dagger+\eta Q^\dagger)} e^{i(S\lambda^\dagger+\lambda S^\dagger)} g_J, \qquad g_0 = e^{-itH} e^{izD} e^{i\omega K}, \quad g_J = e^{i\theta^1 J_1} e^{i\theta^2 J_2} e^{i\theta^3 J_3}. \tag{2.1}$$

In this approach, the coordinates $Z^M = \{t, z, \omega, \eta, \eta^\dagger, \lambda, \lambda^\dagger, \theta^a\}$ in the group manifold will become functions (Goldstone fields) of the worldline parameter $\tau$ –and not superfields [9]–

[3] A different parametrization for the AdS basis is used in [11].

[4] This author employs the non-linear realization approach with a different coset, making use of the geometry of curves to construct the superconformal action [15].

[5] In the following, the index $i$ of the fermionic fields will not be written explicitly.

after the pullback on the world line of the particle is taken. Note that we have also introduced a Goldstone field, $t$, associated to the unbroken translation $H$. Here $g_0$ and $g_J$ parametrize the SO$(2,1)$ and the SU$(2)$ group elements, respectively.

The left-invariant **(LI)** MC one-form $\Omega$ is given by

$$\Omega = -ig^{-1}dg = L^H H + L^D D + L^K K + Q L^{\dagger Q} + L^Q Q^\dagger + S L^{\dagger S} + L^S S^\dagger + L^a J_a = L^A G_A, \quad (2.2)$$

where the one-forms $L^A = dZ^M L_M^A$ are given in appendix A. The MC one-form $\Omega$ satisfies the MC equation

$$d\Omega = -i\Omega \wedge \Omega,$$

which merely asserts that (2.2) defines a flat connection. By definition of the LI MC one forms $L^A$ are invariant under the left action of the group. The explicit form of the infinitesimal group action on the Goldstone fields is constructed in the next section.

## 2.1 Global symmetry

A mechanical system is defined by an action principle, which in this case is given by the integral along the worldline of the pullback of the bosonic LI MC forms. In order to characterize the states of the system, it is necessary to identify the invariances of the action explicitly through the left transformation of the Goldstone fields, $\delta_L Z^M$, under the symmetry group.

As we have introduced all the generators to parametrize the group element $g$, each MC one-form component $L^A$ is invariant under global (rigid) group transformations. The transformation of the Goldstone fields is defined from the left action of the group on $g(Z^M)$ as

$$g(Z^M) \to e^{i\epsilon^A G_A} g(Z^M) = g(Z^M + \delta_L Z^M). \quad (2.3)$$

At the level of the algebra, the left translations $\delta_L Z^M$, are generated by the right-invariant (**RI**) vector fields $\widetilde{V}_B$, dual to the RI MC forms [16],

$$\tilde{\Omega} = -idgg^{-1} = dZ^M R_M{}^A G_A. \quad (2.4)$$

The RI vector fields $\widetilde{V}_B$ are related to the variations of the Goldstone fields $\delta_L Z^M$ through

$$\widetilde{V} = \delta_L Z^M \frac{\partial}{\partial Z^M} = \epsilon^A \, (R^{-1})_A{}^M \frac{\partial}{\partial Z^M} = \epsilon^A \widetilde{V}_A. \quad (2.5)$$

This observation provides an alternative way to construct the $\delta_L Z^M$. From the previous discussion it follows that the bosonic global transformations for $PSU(1,1|2)$ are given by

$$\text{Time translations} : \delta_H t = -\epsilon_H, \quad (2.6)$$

$$\text{Dilatations} : \delta_D t = t\epsilon_D, \quad \delta_D z = \epsilon_D, \quad (2.7)$$

$$\text{Special Conformal} : \delta_K t = -t^2\epsilon_K, \quad \delta_K z = -2t\epsilon_K, \quad \delta_K \omega = e^z \epsilon_K, \quad (2.8)$$

$$\text{SU(2) Rotations} : \delta_{SU(2)}\eta = -\frac{i}{2}\eta\sigma_a\epsilon^a, \; \delta_{SU(2)}\lambda = -\frac{i}{2}\lambda\sigma_a\epsilon^a, \quad (2.9)$$

$$\delta_{SU(2)}\theta^b = (\mathbf{R}^{-1})^b{}_a \epsilon^a. \quad (2.10)$$

The conjugate coordinates $\eta^\dagger$ and $\lambda^\dagger$ transform correspondingly. The matrix $(\mathbf{R}^{-1})^b{}_a$ is given in appendix A and the supersymmetry transformations are in appendix B. In the next section, the action principle is constructed.



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## 3. Dynamics of the superconformal mechanics

The set of LI one-forms obtained from the Lie superalgebra $psu(1,1|2)$ can be used as lagrangians for mechanical systems since they are, by definition, objects that can be integrated along one dimensional trajectories. If we assume an action with the lower number of derivatives,[6] it is naturally given by a general linear combination of the invariant one forms,

$$S = \int b_A \left(L^A\right)^* d\tau, \tag{3.1}$$

where $\left(L^A\right)^*$ stands for the pullback of $L^A$ to the particle's worldline and the $b_A$'s are arbitrary coefficients.

It must be noted here that there is no a priori reason to rule out the fermionic one-forms appropriately multiplied by Grassman numbers in order to obtain the right Grassman parity for a bosonic action. For simplicity, this possibility will not be considered here. The choice of only the bosonic LI MC forms as lagrangians is the first physical assumption in the present construction. Using (A.11)-(A.14), the mechanical model invariant under the $PSU(1,1|\,2)$ group, constructed by taking the pullback along a worldline parameter $\tau$, of a linear combination of the bosonic one-forms $L^H, L^D, L^K, L^a$ reads,

$$\begin{aligned} \mathcal{S} &= \int \mathcal{L}\, d\tau = \int \left(b_H L^H + b_D L^D + b_K L^K + b_a L^a\right)^* \\ &= \int (L^0_K)^* N_K + (L^0_D)^* N_D + (L^0_H)^* N_H + b^a \left(L^0_a\right)^* + N^*_{rest}\,. \end{aligned} \tag{3.2}$$

The coefficients $b_A$ are real but otherwise arbitrary, having the dimensionalities $[b_H] = l^{-1}, [b_K] = l^1, [b_D] = [b_a] = l^0$. The $N_H$, $N_D$, $N_K$ and $N_{res}$ are defined in the appendix A by equations (A.26), (A.27), (A.28) and (A.29), respectively. The SU(2) coset one-forms $L^0_a$ are given in (A.18), and the SO$(1,2)$ coset forms $L^0_K$, $L^0_D$ and $L^0_H$ are given in (A.17).

By inspection of (3.2) it can be noted that the velocity $\dot{\omega}$ appears, up to a boundary term, linearly in the lagrangian and therefore $\omega$ can be eliminated from the action using its own equation of motion,

$$\frac{\delta \mathcal{S}}{\delta \omega} = 0 \quad \Longrightarrow \quad \omega = \frac{-\dot{N}_K - \dot{z} N_K + 2e^{-z}\dot{t} N_D}{2e^{-z}\dot{t} N_K}\,. \tag{3.3}$$

Introducing the new bosonic coordinate $q$, defined by

$$q = \sqrt{2} e^{z/2} \left(\frac{N_K}{b_K}\right)^{1/2}, \tag{3.4}$$

the action (3.2) can now be written as

$$\mathcal{S} = \int d\tau \left[ b_K \frac{\dot{q}^2}{2\dot{t}} - \frac{2\dot{t}}{b_K q^2}(N_H N_K - N_D^2) - \frac{N_D \dot{N}_K}{N_K}\right] + N^*_{rest} + b^a (L^0_a)^*. \tag{3.5}$$

[6]Other combinations can be taken, for instance $\sqrt{b_{AB} L^A_0 L^B_0}$, where $L^A = L^A_0 d\tau$. In general this lagrangian will contains accelerations eventually.

This action clearly resembles the conformal mechanics of [2], with the characteristic $q^{-2}$ potential as the interaction term which produces the nontrivial coupling between bosonic and fermionic degrees of freedom.

One of the relevant aspects found in [1] is the explicit relation between conformal mechanics and a physically nontrivial model describing a charged particle in the near horizon geometry of an extremal, four-dimensional Reissner-Nordström black hole. Indeed, it is trivial to show that the conformal mechanics of [2] [17] describes the motion of a particle on a background isometric to $AdS_2$ . If the particle is charged, however, it would also interact with the electromagnetic field of the black hole, and the trajectory would no longer be a geodesic of the manifold.

In order to compare with ref. [1], it is enlightening to write down the purely bosonic part of the action.

$$\mathcal{S}|_{\eta=\eta^\dagger=\lambda=\lambda^\dagger=0} = \int d\tau \left[ b_K \frac{\dot{q}^2}{2\dot{t}} - \frac{2\dot{t}}{q^2} \left( \frac{b_H b_K - b_D^2}{b_K} \right) \right] + b^a (L_a^0)^*, \tag{3.6}$$

which explicitly reflects the global invariance under the bosonic part of $PSU(1,1|2)$, name - ly, $\mathrm{SO}(1,2) \times \mathrm{SU}(2)$. As $\dot{\theta}^3$ enters linearly in $b_a (L_a^0)^*$, see (A.18), the $\theta^3$ coordinate can be eliminated as well by using its own equation of motion. The resulting action reads

$$\begin{aligned}
\mathcal{S}|_{\eta=\eta^\dagger=\lambda=\lambda^\dagger=0} &= \int d\tau \left[ b_K \frac{\dot{q}^2}{2\dot{t}} - \frac{2\dot{t}}{q^2} \left( \frac{b_H b_K - b_D^2}{b_K} \right) + \sqrt{b_1^2 + b_2^2} \sqrt{\dot{\theta}_1^2 \cos^2\theta_2 + \dot{\theta}_2^2} \right. \\
&\quad \left. - b_3\, \dot{\theta}_1 \sin\theta_2 \right] . \qquad (3.7) \\
&= \int d\tau [L(q) + L(\theta^a)] \qquad (3.8)
\end{aligned}$$

The direct product geometry of the BR metric (1.1) is reflected in the first three terms. They represent a geodesic in $AdS_2$ and a geodesic in $S^2$. The last term can be interpreted, following [18], as the electric coupling of the particle with a monopole field located at the center of $S^2$. Further physical life can be given to this model, comparing $L(q)$ and $L(\theta^a)$ of (3.8) with equation (2.11) of [1] and equation (8.4) of [18] respectively, the constants $b_A$ can be identified as

$$b_K = m \qquad \left(b_H b_K - b_D^2\right) = 2M^2 (m - e)\, m + J^2 \qquad b^a b^a = J^2 \qquad b_3 = ge. \tag{3.9}$$

Here $m$ is the mass, $e$ the electric charge and $J$ is the angular momentum of the particle, while $M$ is the black hole mass and $g$ is the monopole charge. The authors of [1] used the constant angular momentum on shell condition, replacing it in the equation of motion of $q$ and, in advance of quantization, wrote the angular momentum as $l(l+1)$. In the identification (3.9) this convention has not been followed.

The appearance of a monopole field has its roots in the fact that SU(2) is homeomorphic to $S^3$, since an atlas over $S^3$ defines a fiber bundle (the Hopf bundle) classified by the transition function in the $n = 1$ homotopy class of $\pi_1(\mathrm{U}(1)) = Z$. This is identical to the characterization of a magnetic monopole of unit strength.

An interesting mechanism has operated here: the elimination of some non dynamical variables from their equations of motion produced a recombination of the $b^A$'s among themselves, giving rise to the effective parameters of the theory (3.7).

The relation between the parameters of the conformal mechanics and observables have a nice example in the de Alfaro, Fubini and Furlan conformal mechanics [2], where the coefficient $g$ in the hamiltonian

$$H = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) \tag{3.10}$$

can be recognized as the Casimir operator of the conformal group SO$(2,1)$, classifying the irreducible representations of that group [2].

In the action (3.7) there is no direct coupling between the bosonic coordinates $q$ and $\theta^a$; they interact only through the fermions.

## 4. Local Symmetries

### 4.1 Local symmetries in general

In order to examine the local symmetries of the action (3.2) using the NLR approach we followed the procedure developed in [10]. The gauge symmetries are interpreted as right actions on the coset[7].

The general variation of the LI MC one-forms can be described only in terms of the structure constants $\left(f^A{}_{BC}\right)$, the LI MC forms and the variation of the Goldstone fields themselves[8]

$$\delta L^A = d[\delta Z^A] + f^A{}_{BC} L^C \left[\delta Z^B\right], \tag{4.1}$$

where $[\delta Z^A]$ is $L^A$ in which $dZ^M$ is replaced by $\delta Z^M$

$$[\delta Z^A] = \delta Z^M L_M{}^A \qquad \text{for} \qquad L^A = dZ^M L_M{}^A \tag{4.2}$$

The crucial point is the relation between $[\delta Z^A]$ and the right transformation on the group element

$$g(Z^M) \to g(Z^M) e^{i\epsilon^A G_A} = g(Z^M + \delta_R Z^M) \qquad [\delta_R Z^A] = \epsilon^A, \tag{4.3}$$

where $\delta_R Z$ now refers to the right action of the Goldstone field $Z$. After the pullback is taken on the LI MC one-forms, the $\epsilon$ parameter can be made local, $\epsilon \to \epsilon(\tau)$. Using (4.1), the LI MC variations can be computed:

$$\delta_R L^H = d[\delta_R t] + L^D[\delta_R t] - L^H[\delta_R z] + iL^Q[\delta_R \eta^\dagger] - i[\delta_R \eta] L^{Q^\dagger} \tag{4.4}$$

$$\delta_R L^K = d[\delta_R w] - L^D[\delta_R w] + L^K[\delta_R z] + iL^S[\delta_R\ \lambda^\dagger] - i[\delta_R \lambda] L^{S^\dagger} \tag{4.5}$$

$$\delta_R L^D = d[\delta_R z] + 2L^K[\delta_R t] - 2L^H[\delta_R w] + L^Q[\delta_R\ \lambda^\dagger] + [\delta_R \lambda] L^{Q^\dagger} - [\delta_R \eta] L^{S^\dagger} - L^S[\delta_R \eta^\dagger] \tag{4.6}$$

$$\begin{aligned}\delta_R L^a = {} & d[\delta_R \theta^a] + \epsilon^{abc} L^c [\delta_R \theta^b] - i\left(L^Q \sigma^a [\delta_R \lambda^\dagger] - [\delta_R \lambda]\sigma^a L^{Q^\dagger} - [\delta_R \eta]\sigma^a\ L^{S^\dagger}\right) \\ & - iL^S \sigma^a [\delta_R \eta^\dagger]\end{aligned} \tag{4.7}$$

[7] For some earlier work in this direction, see for example [19]

[8] In the case of kappa transformations of superbranes, see for example [20, 21]

The invariance of the action — modulo surface terms — under the above variations requires

$$\begin{pmatrix} b_H & 0 & -b_K \\ 0 & b_H & 2b_D \\ 2b_D & b_K & 0 \end{pmatrix} \begin{pmatrix} [\delta_R t] \\ [\delta_R z] \\ [\delta_R w] \end{pmatrix} = 0 \tag{4.8}$$

$$b^a \epsilon_{abc} [\delta_R \theta^b] = 0 \tag{4.9}$$

$$\Big( [\delta_R \eta] \, , [\delta_R \lambda] \Big) \begin{pmatrix} b_H & -ib_D - b^a \sigma^a, \\ ib_D - b^a \sigma^a & b_K \end{pmatrix} = 0. \tag{4.10}$$

Provided the determinant of the system vanishes, this homogeneous equations have non-trivial solutions for $[\delta_R Z^M]$ corresponding to the different local invariances.

Since the determinant appearing in eq (4.8) vanishes, there is a non-trivial solution

$$[\delta_R t] = \epsilon(\tau), \quad [\delta_R z] = -2\frac{b_D}{b_K}\epsilon(\tau), \quad [\delta_R w] = \frac{b_H}{b_K}\epsilon(\tau), \quad \text{and others} = 0, \tag{4.11}$$

where $\epsilon(\tau)$ is an arbitrary function. We will refer to this transformation as $T$ symmetry.

The (4.9) is the local U(1) transformation

$$[\delta_R \theta^a] = b^a \alpha(\tau), \qquad \text{and others} = 0, \tag{4.12}$$

where $\alpha(\tau)$ is an arbitrary function.

The $T$ and U(1) symmetries are present for any non-vanishing value of the coupling constants. This implies that the number of physical bosonic degrees of freedom described by the action (3.2) is two, therefore it is not describing the most general motion of the test particle in the near horizon of geometry of a $\mathcal{N} = 2$ charged four-dimensional extremal black hole, that has three bosonic degrees of freedom.

The number of linearly realized worldline supersymmetries of the lagrangian is related to the rank of the matrix in (4.10). When $b_H b_K \neq b_D^2 + b^a b^a$ the $4 \times 4$ matrix in (4.10) has the maximal rank and (4.10) only has trivial solution $[\delta_R \eta] = [\delta_R \lambda] = 0$. In this case the system has no local fermionic symmetry and all supersymmetries are broken (non-BPS particle).

If

$$b_H b_K - b_D^2 = b^a b^a, \tag{4.13}$$

the rank of the matrix (4.10) is 2 and the number of linearly realized supersymmetries of the worldline is 4 (BPS particle). This relation implies the equality between the Casimir invariants of the SU(2) and SO(1,2) sectors.

The action acquires a new local symmetry, the so-called $\kappa$ symmetry. The corresponding non-trivial solution is

$$[\delta_R \eta] = \kappa_\eta(\tau), \qquad [\delta_R \lambda] = \kappa_\eta(\tau)(\frac{ib_D}{b_K} + \frac{b^a \sigma_a}{b_K}), \tag{4.14}$$

and other bosonic $[\delta_R Z^A]$ are zero,

$$[\delta_R t] = [\delta_R z] = [\delta_R \omega] = [\delta_R \theta^a] = 0, \tag{4.15}$$

where $\kappa^i_\eta(\tau)$ is a SU(2) doublet arbitrary Grassman-valued function of $\tau$.

Following [10] we can construct the generators of the local algebra. In our context the local symmetries $T$, U(1) and $\kappa$, are

$$T = H - 2\frac{b_D}{b_K}D + \frac{b_H}{b_K}K - 2\frac{b^a}{b_K}J_a, \tag{4.16}$$

$$B = b^a J_a, \tag{4.17}$$

$$\tilde{Q}^i = Q^i + S^j\left(-\frac{ib_D}{b_K}\delta_j{}^i + \frac{b^a}{b_K}(\sigma_a)_j{}^i\right) \tag{4.18}$$

$$\tilde{Q}^\dagger_i = Q^\dagger_i + \left(\frac{ib_D}{b_K}\delta_i{}^j + \frac{b^a}{b_K}(\sigma_a)_i{}^j\right)S^\dagger_j. \tag{4.19}$$

In the case of (4.13) they generate unbroken symmetry of the Lagrangian (3.2) and form a subalgebra of the $psu(1,1|2)$,

$$\left[\tilde{Q}^i, \tilde{Q}^\dagger_j\right]_+ = \delta^i{}_j T, \qquad \left[\tilde{Q}^i, \tilde{Q}^j\right]_+ = \left[\tilde{Q}^i, T\right] = 0, \tag{4.20}$$

$$\left[B, \tilde{Q}^i\right] = \frac{1}{2}\tilde{Q}^j(b^a\sigma_a)_j{}^i, \qquad [B, T] = 0. \tag{4.21}$$

The diffeomorphism invariance, $\tau \to \tau'(\tau)$ is not independent of the local symmetries previously discussed. Moreover, when the condition (4.13) for $\kappa$ symmetry is satisfied, diffeomorphims are equivalent to linear combinations of the local symmetries obtained from the right translations, with parameters chosen in terms of $\delta\tau = \varepsilon(\tau)$ as

$$\epsilon(\tau) = (L^H)^*\varepsilon(\tau), \qquad \alpha(\tau) = \frac{(b^b L^b)^*}{b^c b^c}\varepsilon(\tau), \qquad \kappa_\eta(\tau)s(\theta) = (L^Q)^*\varepsilon(\tau). \tag{4.22}$$

In the non-BPS case there is no kappa transformation.

In the appendix C it is shown that these combinations of the local transformations and diffeomorphisms differ by trivial variations, *i.e.* (graded)anti-symmetric combinations of the equations of motion.

### 4.2 Kappa symmetry

It has been shown that if the constants of the Lagrangian satisfy (4.13), the action is invariant under the kappa transformations. The transformation of the fields around the configuration $\eta = \eta^\dagger = 0$,[9] is obtained from (4.14) and (4.15) as

$$\delta_\kappa\eta|_{\eta=\eta^\dagger=0} = \kappa_\eta s(\theta)^{-1}, \qquad \delta_\kappa\eta^\dagger|_{\eta=\eta^\dagger=0} = s(\theta)\kappa_{\eta^\dagger}. \tag{4.23}$$

Where $s(\theta)$ is the spin one half representation of the SU(2) group, by redefinition of the parameter $\kappa$, $s(\theta)$ can be reabsorbed. Then it follows that in any neighborhood of $\eta = \eta^\dagger = 0$ the gauge slice

$$\eta = \eta^\dagger = 0 \tag{4.24}$$

is accessible. In this gauge the remaining coordinates transform as

$$\delta_\kappa\lambda|_{\eta=\eta^\dagger=0} = \kappa_\eta - \frac{1}{2}\kappa_\eta(\lambda\lambda^\dagger) - \frac{1}{2}\lambda(\kappa_\eta\lambda^\dagger - \lambda\kappa_{\eta^\dagger}) \tag{4.25}$$

[9]The transformation for a general configuration is rather complicated. We give it in the $OSP(2|2)$ case in the appendix D.

$$\delta_\kappa t|_{\eta=\eta^\dagger=0} = 0 \tag{4.26}$$

$$\delta_\kappa z|_{\eta=\eta^\dagger=0} = -(\lambda\delta_\kappa\eta^\dagger + \delta_\kappa\eta\lambda^\dagger) \tag{4.27}$$

$$\delta_\kappa\omega|_{\eta=\eta^\dagger=0} = -\omega(\lambda\delta_\kappa\eta^\dagger + \delta_\kappa\eta\lambda^\dagger) + \frac{i}{2}(\lambda\delta_\kappa\lambda^\dagger - \delta_\kappa\lambda\lambda^\dagger) + \frac{i}{2}(\lambda\delta_\kappa\eta^\dagger - \delta_\kappa\eta\lambda^\dagger)(\lambda\lambda^\dagger) \tag{4.28}$$

$$\delta_\kappa\theta^a|_{\eta=\eta^\dagger=0} = -i(\lambda\sigma_b\delta_\kappa\eta^\dagger - \delta_\kappa\eta\sigma_b\lambda^\dagger)\left(\mathbf{R}_b^a\right)^{-1} \tag{4.29}$$

As can be seen from the previous results, when the kappa condition (4.13) is satisfied, it is possible to gauge away half of the fermions. In the next section diffeomorphism and kappa symmetry are further fixed, residual transformations found and BPS states obtained.

### 4.3 Gauge fixed lagrangian and residual global transformations

The kappa symmetry can be used to further simplify the form of the lagrangian. In fact, imposing (4.13), and setting $\eta = \eta^\dagger = 0$ and the static gauge $t = \tau$, the action (3.5) becomes

$$\mathcal{S} = \int dt\, b_K \left[\frac{\dot{q}^2}{2} - \frac{2}{q^2}\left(\frac{1}{4}(\lambda\lambda^\dagger)^2 - (\lambda\sigma_a\lambda^\dagger)\mathcal{S}_{ab}\frac{b_b}{b_K} + \frac{b_a b_a}{b_K^2}\right) - \frac{i}{2}(\lambda\dot{\lambda}^\dagger - \dot{\lambda}\lambda^\dagger)\right] + \left(L_a^0\right)^* b_a, \tag{4.30}$$

where, in this gauge,

$$q = \sqrt{2}e^{\frac{z}{2}}. \tag{4.31}$$

Moreover the coupling constant of $q^{-2}$ computed in [1] for the kappa-symmetric particle $(e = m)$ is exactly reproduced.

$$\frac{g}{2} = 4\frac{b_a b_a}{b_K} = 4\frac{J^2}{m} \tag{4.32}$$

As it was previously pointed out, $\theta^3$ is non dynamical, its elimination reduces (4.30) to

$$\begin{aligned}\mathcal{L} = {} & b_K\frac{\dot{q}^2}{2} - \frac{2}{q^2 b_K}\left(b_K^2\frac{1}{4}(\lambda\lambda^\dagger)^2 + \frac{b_a b_a}{b_K^2}\right) - \frac{i}{2}b_K(\lambda\dot{\lambda}^\dagger - \dot{\lambda}\lambda^\dagger) \\ & + \sqrt{b_1^2 + b_2^2}\sqrt{\left(\dot{\theta}_1\cos\theta_2 + j\cdot e_1\right)^2 + \left(\dot{\theta}_2 + j\cdot e_2\right)^2} \\ & + b_3\left(-\dot{\theta}_1\sin\theta_2 + j\cdot e_3\right),\end{aligned} \tag{4.33}$$

where $j_a = 2\frac{(\lambda\sigma_a\lambda^\dagger)}{q^2}$ and the orthonormal basis $e_a$ is given by

$$e_1 = \begin{pmatrix}\cos\theta_2 \\ \sin\theta_1\sin\theta_2 \\ \cos\theta_1\sin\theta_2\end{pmatrix} \qquad e_2 = \begin{pmatrix}0 \\ \cos\theta_1 \\ -\sin\theta_1\end{pmatrix} \qquad e_3 = \begin{pmatrix}-\sin\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2\end{pmatrix}. \tag{4.34}$$

It must be noted that action (4.33) still has the U(1) gauge invariance (4.12), $\delta\theta^b = \alpha(\tau)b_a\left(\mathbf{L}_a^b\right)^{-1}$, which after $\theta_3$ is eliminated becomes

$$\delta\theta_1 = \tilde{\alpha}(\tau)(\dot{\theta}_1 + j\cdot e_1/\cos\theta_2), \qquad \delta\theta_2 = \tilde{\alpha}(\tau)(\dot{\theta}_2 + j\cdot e_2), \tag{4.35}$$

where

$$\tilde{\alpha}(\tau)=\frac{\sqrt{b_1^2+b_2^2}}{\sqrt{(\dot{\theta}_1+j\cdot e_2/\cos\theta_2)^2+(\dot{\theta}_2+j\cdot e_2)^2}}\alpha(\tau), \tag{4.36}$$

and $\delta Z^M=0$ for other fields.

The gauge fixing has changed the form of the global transformations. This is because local transformations must be used in order to respect the gauge slice previously chosen. This means that local compensating transformations must be introduced. It is straightforward to show that they are given by

$$\delta^*\left(t-\tau\right)|_{\eta=\eta^\dagger=0;t=\tau}=\left(\varepsilon\dot{t}+\delta_\kappa t+\delta_G t\right)|_{\eta=\eta^\dagger=0;t=\tau}=0\Longrightarrow\varepsilon=-\delta_G t, \tag{4.37}$$

$$\delta^*\eta|_{\eta=\eta^\dagger=0;t=\tau}=\left(\varepsilon\dot{\eta}+\delta_\kappa\eta+\delta_G\eta\right)|_{\eta=\eta^\dagger=0;t=\tau}=0\Longrightarrow\kappa_\eta=-\delta_G\eta \tag{4.38}$$

$$\delta^*\eta^\dagger|_{\eta=\eta^\dagger=0;t=\tau}=\left(\varepsilon\dot{\eta}^\dagger+\delta_\kappa\eta^\dagger+\delta_G\eta^\dagger\right)|_{\eta=\eta^\dagger=0;t=\tau}=0\Longrightarrow\kappa_{\eta^\dagger}=-\delta_G\eta^\dagger, \tag{4.39}$$

where $\delta_G$ stands for any global $\mathrm{SU}(1,1|\,2)$ transformation. The residual transformations for the remaining coordinates $\delta^*$ are defined in the same way as the former variations, but with the local parameters given by (4.37), (4.39),

$$\delta^*_H q=\epsilon_H\dot{q}\qquad\delta^*_H\lambda=\epsilon_H\dot{\lambda}\qquad\delta^*_H\theta^a=\epsilon_H\dot{\theta}^a \tag{4.40}$$

$$\delta^*_D q=\epsilon_D\left(\frac{q}{2}-t\dot{q}\right)\qquad\delta^*_D\lambda=-t\epsilon_D\dot{\lambda}\qquad\delta^*_D\theta^a=-t\epsilon_D\dot{\theta}^a \tag{4.41}$$

$$\delta^*_K q=-t\epsilon_K q+t^2\epsilon_K\dot{q}\qquad\delta^*_K\lambda=t^2\epsilon_K\dot{\lambda}\qquad\delta^*_K\theta^a=t^2\epsilon_K\dot{\theta}^a \tag{4.42}$$

$$\delta^*_a q=0\qquad\delta^*_a\lambda=-\frac{i}{2}\lambda\sigma_a\epsilon\qquad\delta_a\theta^b=\mathbf{R}^{-1}_{ab}\epsilon \tag{4.43}$$

$$\delta^*_{Q_i}q=\frac{1}{\sqrt{2}}\epsilon_Q\lambda^\dagger\qquad\delta^*_{Q_i}\lambda^\dagger_k=\frac{1}{\sqrt{2}q}\lambda^\dagger_k\lambda^\dagger_i\epsilon_Q\qquad\delta^*_{Q_i}\theta^a=-i\epsilon_Q\left(\sigma_b\lambda^\dagger\right)_i\mathbf{R}^{-1}_{ba}\frac{\sqrt{2}}{q}$$

$$\delta^*_{Q_i}\lambda_k=\left(i\left(\frac{-\dot{\tilde{q}}q}{2}\right)\delta_{ik}-\frac{b_a}{b_K}\left(\sigma_b\right)_i^k\mathcal{S}_{ba}+\frac{1}{2}\delta_{ik}\lambda\lambda^\dagger-\frac{1}{2}\lambda_k\lambda^\dagger_i\right)\epsilon_Q\frac{\sqrt{2}}{q} \tag{4.44}$$

$$\delta^*_{S_i}q=it\frac{1}{\sqrt{2}}\lambda^\dagger_i\epsilon_S\qquad\delta^*_{S_i}\lambda^\dagger_k=-\frac{it}{\sqrt{2}q}\lambda^\dagger_k\lambda^\dagger_i\epsilon_S\qquad\delta^*_{S_i}\theta^a=-\frac{t\sqrt{2}}{q}\epsilon_S\left(\sigma_b\lambda^\dagger\right)^i\mathbf{R}^{-1}_{ba}$$

$$\delta^*_{S_i}\lambda_k=\left(\left(\frac{q^2}{2}-\frac{\dot{q}q}{2}t\right)\delta_{ik}+it\frac{b_a}{b_K}\left(\sigma_b\right)_i^k\mathcal{S}_{ba}-it\frac{1}{2}\delta_{ik}(\lambda\lambda^\dagger)+\frac{1}{2}it\lambda_k\lambda^\dagger_i)\right)\frac{\sqrt{2}}{q}\epsilon_S \tag{4.45}$$

To study the existence of BPS states, the residual transformation of the fermions under $Q$'s and $S$'s are considered. In this way, two BPS equations arise:

$$\delta^*_{Q_i}\lambda_k=\delta^*_{Q_i^\dagger}\lambda^\dagger_k=0\Longrightarrow\left(\frac{\dot{q}q}{2}\right)^2+\frac{b_a b_a}{b_K^2}=0, \tag{4.46}$$

$$\delta^*_{S_i}\lambda_k=\delta^*_{S_i^\dagger}\lambda^\dagger_k=0\Longrightarrow\left(\frac{q^2}{2}-t\frac{\dot{q}q}{2}\right)^2+t^2\frac{b_a b_a}{b_K^2}=0, \tag{4.47}$$

As both of them are the sum of two positive terms, a necessary condition for the existence of BPS states is $b_a=0$. Then, by eq. (4.32), the coupling constant vanishes and the system is just the free particle. Then, the $\dot{q} = 0$ configuration, (4.46), saturates the bound of the free particle Hamiltonian $H$, meanwhile the $\dot{q} = \frac{q}{t}$ configuration, (4.47), saturates the bound of the free particle special conformal transformation generator $K$. So the existence of non trivial 1/2 BPS states is ruled out.

### 4.4 Bosonic Motions

Let us now study the bosonic trajectories of our model. If we fix the diffeomorphism by taking the gauge $\dot{t} = 1$ the Lagrangian (3.7) becomes

$$\mathcal{S} = \int d\tau \left[ m\,\frac{\dot{q}^2}{2} - \frac{2}{mq^2}\left(b_H b_K - b_D^2\right) + \sqrt{b_1^2 + b_2^2}\sqrt{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2} + eg\,\left(-\dot{\phi}\cos\theta\right)\right], \tag{4.48}$$

where $\phi = \theta_1,\ \theta = \frac{\pi}{2} - \theta_2$. The first class constraint associated to the U(1) gauge invariance of the Lagrangian is,

$$\Psi = \frac{1}{2}\left[p_\theta^2 + \left(\frac{p_\phi + eg\cos\theta}{\sin\theta}\right)^2 - (b_1^2 + b_2^2)\right] \sim 0, \tag{4.49}$$

where $p_q, p_\theta, p_\phi$ are the canonical momenta associated to the coordinates $q, \theta, \phi$. The Dirac Hamiltonian is

$$H = \frac{p_q^2}{2m} + \frac{2}{q^2}\left(\frac{b_H b_K - b_D^2}{b_K}\right) + \Lambda\Psi, \tag{4.50}$$

where $\Lambda$ is an arbitrary function of $t$. In the presence of a monopole background the conserved angular momentum is

$$\mathbf{J} = p_\theta \mathbf{e}_\phi - \left(\frac{p_\phi + eg\cos\theta}{\sin\theta}\right)\mathbf{e}_\theta - (eg)\mathbf{e}_r, \tag{4.51}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ are the othonormal unit vectors in the polar coordinates. The constraint (4.49) means that the value of $\mathbf{J}^2$ is fixed by parameters of the lagrangian as

$$\mathbf{J}^2 = p_\theta^2 + \left(\frac{p_\phi + eg\cos\theta}{\sin\theta}\right)^2 + (eg)^2 \sim b_a^2. \tag{4.52}$$

We have considered the kappa invariant case when (4.13) is satisfied. Furthermore, in a gauge where the arbitrary function $\Lambda$ appears as

$$\Lambda = \frac{4}{mq^2} \tag{4.53}$$

the Hamiltonian becomes

$$H^* = \frac{p_q^2}{2m} + \frac{2}{mq^2}\,\mathbf{J}^2 = \frac{p_q^2}{2m} + \frac{2}{mq^2}\left(p_\theta^2 + (\frac{p_\phi + eg\cos\theta}{\sin\theta})^2 + (eg)^2\right). \tag{4.54}$$

The corresponding lagrangian is

$$L^* = \frac{m}{2}\dot{q}^2 + \frac{mq^2}{8}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) - 2\frac{(eg)^2}{mq^2} - (eg)\dot{\phi}\cos\theta. \tag{4.55}$$

The Lagrangian (4.55) agrees with the bosonic part of the $D(2,1,\alpha=-1)$ superconformal lagrangian considered in [11]. It coincides also with that of the D0 particle on a black hole attractor [8] up to second order expansion in derivatives.

Although the form of the lagrangians coincides, the physical content of these models is different. The lagrangian (4.55) has associated the constraint (4.49);

$$\Psi^* = \frac{1}{2}\left[(\frac{mq^2}{4})^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)-(b_1^2+b_2^2)\right]\sim 0. \tag{4.56}$$

In both cases the trajectories of the particle are on a two dimensional cone. However, in our case the total angular momentum squared is constrained by (4.56), it follows that in terms of the parameters of our lagrangian the opening angle of the cone is fixed as $\tan^{-1}(\frac{\sqrt{b_1^2+b_2^2}}{b_3})$.

## 5. Covariant AdS parametrization

At the quantum level, the conformal mechanics has no ground state associated to the hamiltonian $H$. The wave function spreads out to spatial infinity. The authors of [2] suggest that one should consider the eigenstates of the compact operator $P_0=\frac{1}{2}(H+K)$ which has a discrete spectrum of normalizable eigenstates. From the perspective of the particle motion near the black hole it corresponds to a different choice of time [1]. In fact the conjugate variable to $P_0$ is the global time of $AdS_2$ and can describe the motion of the particle entering through the horizon, instead the time conjugate to $H$ only describe the motion of the particle outside of the horizon. Therefore it is also natural to study the dynamics of the superconformal particle using the new basis, that we call *AdS basis*. In our approach this implies a new parametrization of the coset, that we take

$$g \;=\; g_0^{AdS_2}\; g_0^{S^2}\; e^{i(Q\bar{\eta}+\eta\bar{Q})}\; e^{i(S\bar{\lambda}+\lambda\bar{S})}\; e^{iM_{01}y}\; e^{iJ_3y'}, \tag{5.1}$$

where

$$g_0^{AdS_2}=e^{iP_0x^0}e^{iP_1x^1}, \qquad g_0^{S^2}=e^{iJ_1\theta^1}e^{iJ_2\theta^2} \tag{5.2}$$

and the $AdS_2$ generators $P_0, P_1, M_{01}$ are related to the conformal ones by

$$P_0=\frac{H+K}{2}, \qquad P_1=D, \qquad M_{01}=\frac{H-K}{2}. \tag{5.3}$$

The MC one form is

$$\Omega = L^{P_\mu}P_\mu+L^{M_{01}}M_{01}+L^{J_a}J_a+L^{J_3}J_3+QL^{\dagger Q}+L^QQ^\dagger+SL^{\dagger S}+L^SS^\dagger. \tag{5.4}$$

where $\mu=0,1$ and $a=1,2$. The invariant particle Lagrangian is a sum of bosonic forms

$$\mathcal{L}=L^{P_\mu}b_{P_\mu}+L^{M_{01}}b_{M_{01}}+L^{J_a}b_a+L^{J_3}b_3. \tag{5.5}$$

In (5.1) we have put $e^{iM_{01}y}$, $e^{iJ_{01}y'}$ at the right so that $dy$ and $dy'$ terms appear in the lagrangian in total derivative forms and can be omitted. The Lagrangian is written as

$$\mathcal{L}=(A\;\sinh y+B\;\cosh y+C)+(A'\;\sin y'+B'\;\cos y'+C'). \tag{5.6}$$

where

$$
\begin{aligned}
A &= b_{P_1} L_1^{P_0} + b_{P_0} L_1^{P_1}, \qquad B = b_{P_0} L_1^{P_0} + b_{P_1} L_1^{P_1}, \qquad C = b_{M_{01}} L_1^{M_{01}} \\
A' &= b_2 L_1^{J_1} - b_1 L_1^{J_2}, \qquad B' = b_1 L_1^{J_1} + b_2 L_1^{J_2}, \qquad C' = b_3 L_1^{J_3}
\end{aligned} \tag{5.7}
$$

where the explicit forms of $L_1^A$'s are given in the appendix A. The Goldstone fields $y$ and $y'$ are non-dynamical variables and can be eliminated using their equations of motion and

$$
\begin{aligned}
\mathcal{L} = &-\sqrt{b_{P_0}^2 - b_{P_1}^2}\,\sqrt{(L_1^{P_0})^2 - (L_1^{P_1})^2} + b_{M_{01}}\, L_1^{M_{01}} \\
&-\sqrt{b_1^2 + b_2^2}\,\sqrt{(L_1^{J_1})^2 + (L_1^{J_2})^2} + b_3\, L_1^{J_3}.
\end{aligned} \tag{5.8}
$$

As the previous discussions in section 4 the action from (5.8) is invariant under two bosonic local symmetries, diffeomorphism and U(1). It is also invariant under the kappa symmetry if the coefficients of the Lagrangian are verifying

$$
b_{P_0}^2 - b_{P_1}^2 - b_{M_{01}}^2 = b_a b_a. \tag{5.9}
$$

which is corresponding to (4.13), $b_H b_K - b_D^2 = b_a b_a$.

As in the *conformal* basis this relation implies and equality between the Casimir invariants of SU(2) and SU(1,1). The two WZ terms represents the coupling to the electromganetic field.

The lagrangian (5.8), where the fermions have been set to zero,

$$
\begin{aligned}
\mathcal{L} = &-\sqrt{b_{P_0}^2 - b_{P_1}^2}\,\sqrt{(dx^0 \cosh\frac{x^1}{R})^2 - (dx^1)^2} + b_{M_{01}}\,\frac{dx^0}{R}\sinh\frac{x^1}{R} \\
&-\sqrt{b_1^2 + b_2^2}\,\sqrt{(d\theta^1 \cos\theta^2)^2 + (d\theta^2)^2} - b_3\, d\theta^1 \sin\theta^2,
\end{aligned} \tag{5.10}
$$

does not reproduce the motion of a relativistic particle in $AdS_2 \times S_2$, because the lagrangian here has two square roots which is not equivalent to the lagrangian studied in references [13], [14, 8, 15]. The two systems have different numbers of degrees of freedom since they possess different bosonic gauge symmetries. A similar effect occurs in the conformal basis due to the appearance of two gauge symmetries, diffeomorphisms and U(1) transformations. In the $D0$ brane lagrangian, instead, there are only diffeomorphisms. Since we have interpreted the gauge transformations as induced by the right action on the coset by unbroken translation [10], it means that there are two unbroken translations given by $P_0$ and the $b^a J_a$ in the present case.

## 6. Discussion and Outlook

The BPS and non BPS dynamics of a superconformal particle has been constructed, using *only* the method of non-linear realization without resorting to superfields or requiring further constraints [9]. The coset $PSU(1,1|2)$ had been considered, as in [5, 6]. The particle action contains six couplings constants and is invariant under two set of bosonic gauge symmetries, diffeomorphisms and U(1) gauge transformations. When the condition on the coupling constants (4.13) is verified, the action becomes also kappa symmetric. This relation implies the equality between the Casimir operators of the SU(2) and the SU(1, 1) sectors. Following reference [10] these gauge symmetries can be interpreted as generated by the unbroken "translations" via the right action. Furthermore, the algebra verified by the generators of gauge transformations was found.

The description of the dynamics has been done in two different bases or parametrizations of the coset: the conformal basis and the AdS basis. In both cases the kappa-symmetric and non kappa-symmetric models can be viewed as describing the equatorial motion of a particle near the horizon of a $\mathcal{N} = 2$ charged four-dimensional extremal black hole. It turns that the particle has its total angular momentun squared fixed, this value is determined by the parameters appearing in the lagrangian. They are not describing the entire three dimensional dynamics of the $D0$ particle.

The analysis of the existence of BPS states shows trough equations (4.46) and (4.47) that they only exist in a highly degenerate case of the conformal mechanics, namely, in the free particle case. A natural question then arises as to whether it is possible to obtain the lagrangian of a $D0$ brane from the method of non-linear realization without any extra geometrical or physical requirements. This point will be addressed in a future study.

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## A. Notation and conventions

The $psu(1,1|2)$ algebra

$$[H,D] = iH, \qquad [K,D] = -iK, \qquad [H,K] = 2iD, \tag{A.1}$$

$$[J_a, J_b] = i\varepsilon_{abc}J_c, \tag{A.2}$$

$$\left[Q^i, Q^\dagger_j\right]_+ = \delta^i{}_j H, \qquad \left[S^i, S^\dagger{}_j\right]_+ = \delta^i{}_j K, \tag{A.3}$$

$$\left[S^\dagger{}_i, Q^j\right]_+ = -(\sigma_a)_i{}^j J_a + i\delta_j{}^i D, \qquad \left[Q^\dagger{}_i, S^j\right]_+ = -(\sigma_a)_i{}^j J_a - i\delta_j{}^i D, \tag{A.4}$$

$$\left[D, Q^i\right] = -\frac{i}{2}Q^i, \qquad \left[D, {Q^\dagger}_i\right] = -\frac{i}{2}{Q^\dagger}_i, \tag{A.5}$$
$$\left[D, S^i\right] = \frac{i}{2}S^i, \qquad \left[D, {S^\dagger}_i\right] = \frac{i}{2}{S^\dagger}_i, \tag{A.6}$$
$$\left[K, Q^i\right] = S^i, \qquad \left[K, {Q^\dagger}_i\right] = -{S^\dagger}_i, \tag{A.7}$$
$$\left[H, S^i\right] = Q^i, \qquad \left[H, {S^\dagger}_i\right] = -{Q^\dagger}_i \tag{A.8}$$
$$\left[J_a, Q^i\right] = \frac{1}{2}Q^j(\sigma_a)_j{}^i, \qquad \left[J_a, {Q^\dagger}_i\right] = -\frac{1}{2}(\sigma_a)_i{}^j{Q^\dagger}_j, \tag{A.9}$$
$$\left[J_a, S^i\right] = \frac{1}{2}S^j(\sigma_a)_j{}^i, \qquad \left[J_a, {S^\dagger}_i\right] = -\frac{1}{2}(\sigma_a)_i{}^j{S^\dagger}_j. \tag{A.10}$$

**Maurer-Cartan forms**

The Maurer-Cartan one-forms are explicitly given by

$$L^H = L^0_H + \frac{1}{4}L^0_K(\eta\eta^\dagger)^2 - \frac{i}{2}(\eta d\eta^\dagger - d\eta\eta^\dagger), \tag{A.11}$$
$$L^D = L^0_D\{1 + \frac{1}{2}(\lambda\eta^\dagger + \eta\lambda^\dagger)\} + \frac{i}{2}L^0_K(\eta\eta^\dagger)(\lambda\eta^\dagger - \eta\lambda^\dagger) + (\lambda d\eta^\dagger + d\eta\lambda^\dagger), \tag{A.12}$$
$$\begin{aligned} L^K =\ & L^0_K\{1 + (\lambda\eta^\dagger + \eta\lambda^\dagger) - \frac{1}{2}(\eta\sigma_a\eta^\dagger)(\lambda\sigma_a\lambda^\dagger) + \frac{1}{4}(\eta\eta^\dagger)(\lambda\lambda^\dagger)(\lambda\eta^\dagger + \eta\lambda^\dagger)\} + \frac{1}{4}L_H(\lambda\lambda^\dagger)^2 \\ & -\frac{i}{4}L^0_D(\lambda\eta^\dagger - \eta\lambda^\dagger)(\lambda\lambda^\dagger) - \frac{i}{2}(\lambda d\lambda^\dagger - d\lambda\lambda^\dagger) - \frac{i}{2}(\lambda d\eta^\dagger - d\eta\lambda^\dagger)(\lambda\lambda^\dagger), \end{aligned} \tag{A.13}$$
$$\begin{aligned} L^{J_b} =\ & L^0_{J_b} + \Big[i(\lambda\sigma_a d\eta^\dagger - d\eta\sigma_a\lambda^\dagger) + \frac{i}{2}L^0_D\left(\lambda\sigma_a\eta^\dagger - \eta\sigma_a\lambda^\dagger\right) \\ & +L^0_K\{-(\eta\sigma_a\eta^\dagger) - \frac{1}{2}(\eta\eta^\dagger)\left(\lambda\sigma_a\eta^\dagger + \eta\sigma_a\lambda^\dagger\right)\} - L_H(\lambda\sigma_a\lambda^\dagger)\Big]\mathcal{S}_{ab}(\theta), \end{aligned} \tag{A.14}$$
$$L^Q = \left[d\eta + \frac{1}{2}\,L^0_D\eta - iL_H\lambda - \frac{i}{2}L^0_K(\eta\eta^\dagger)\eta\right]s(\theta), \tag{A.15}$$
$$\begin{aligned} L^S =\ & \Big[d\lambda + \frac{1}{2}d\eta(\lambda\lambda^\dagger) - \lambda(\lambda d\eta^\dagger) - L_H\frac{i}{2}(\lambda\lambda^\dagger)\lambda - L^0_D\frac{1}{2}\left(\lambda + (\lambda\eta^\dagger)\lambda - \frac{1}{2}\eta(\lambda\lambda^\dagger)\right) \\ & +L^0_K\{-i\eta + (\eta\sigma_a\eta^\dagger)\frac{i}{2}\lambda\sigma_a - \frac{i}{2}(\eta\eta^\dagger)(\lambda\eta^\dagger)\lambda - \frac{i}{4}\eta(\eta\eta^\dagger)(\lambda\lambda^\dagger)\}\Big]\,s(\theta). \end{aligned} \tag{A.16}$$

$L^{Q^\dagger}$ and $L^{S^\dagger}$ are conjugate to $L^Q$ and $L^S$ respectively. $L^0_{H,D,K}$ are the Maurer-Cartan forms associated to the SO(1,2),

$$L^0_H = -e^{-z}dt, \qquad L^0_D = 2e^{-z}\omega dt + dz, \qquad L^0_K = -e^{-z}\omega^2 dt - \omega dz + d\omega \tag{A.17}$$

while those of the SU(2) are

$$L^0_a = d\theta^b\,\mathbf{L}_{ba}, \qquad \mathbf{L}_{ba} = \begin{pmatrix} \cos\theta^2\cos\theta^3 & \cos\theta^2\sin\theta^3 & -\sin\theta^2 \\ -\sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.18}$$

$s(\theta)$ and $\mathcal{S}_{ab}(\theta)$ are spinor and adjoint representations of the SU(2) rotation $e^{i\theta^a J_a}$ given in (A.20) and (A.21) in the appendix A respectively.

**SU(2) matrices**

The group element $g_J$ in the SU(2) sector (2.1) is

$$g_J = e^{i\theta^1 J_1} e^{i\theta^2 J_2} e^{i\theta^3 J_3} \tag{A.19}$$

$\mathcal{S}_{ab}$ is the adjoint representation of the $g_J$

$$\mathcal{S}_{ab} = \begin{pmatrix} \cos\theta^2\cos\theta^3 & \cos\theta^2\sin\theta^3 & -\sin\theta^2 \\ \sin\theta^1\sin\theta^2\cos\theta^3 - \cos\theta^1\sin\theta^3 & \sin\theta^1\sin\theta^2\sin\theta^3 + \cos\theta^1\cos\theta^3 & \sin\theta^1\cos\theta^2 \\ \cos\theta^1\sin\theta^2\cos\theta^3 + \sin\theta^1\sin\theta^3 & \cos\theta^1\sin\theta^2\sin\theta^3 - \sin\theta^1\cos\theta^3 & \cos\theta^1\cos\theta^2 \end{pmatrix} \tag{A.20}$$

while the spinorial representation is:

$$s = e^{i\frac{\theta^1}{2}\sigma_1} e^{i\frac{\theta^2}{2}\sigma_2} e^{i\frac{\theta^3}{2}\sigma_3}. \tag{A.21}$$

It holds

$$s^\dagger\sigma_a s = \mathcal{S}_{ab}\sigma_b, \qquad (\mathcal{S}^T)_{ad} d\mathcal{S}_{db} = \epsilon_{abc} L_c^0, \qquad s^\dagger ds = \frac{i}{2} L_c^0 \sigma_c. \tag{A.22}$$

The SU(2) left invariant one forms (A.18) are

$$L^0_{J_a} = d\theta^b\, \mathbf{L}_{ba}, \qquad \mathbf{L} = \begin{pmatrix} \cos\theta^2\cos\theta^3 & \cos\theta^2\sin\theta^3 & -\sin\theta^2 \\ -\sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.23}$$

The right invariant one forms defined by $-idg_J g_J^{-1} = J_a R^0_{J_a}$ are

$$R^0_{J_a} = d\theta^b\, \mathbf{R}_{ba}, \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^1 & -\sin\theta^1 \\ -\sin\theta^2 & -\sin\theta^1\cos\theta^2 & \cos\theta^1\cos\theta^2 \end{pmatrix}. \tag{A.24}$$

The matrix $\mathbf{R}^{-1}_{ab}$ is the inverse of $\mathbf{R}$;

$$\mathbf{R}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \sin\theta^1\tan\theta^2 & \cos\theta^1 & \sin\theta^1/\cos\theta^2 \\ \cos\theta^1\tan\theta^2 & -\sin\theta^1 & \cos\theta^1/\cos\theta^2 \end{pmatrix} = \mathcal{S}\,\mathbf{L}^{-1}. \tag{A.25}$$

When the lagrangian was constructed in (3.5), the following shorthands are used

$$N_H = b_H + b_K\frac{1}{4}(\lambda\lambda^\dagger)^2 - (\lambda\sigma_a\lambda^\dagger)\mathcal{S}_{ab}b_b \tag{A.26}$$

$$N_D = b_D\{1+\frac{1}{2}(\lambda\eta^\dagger+\eta\lambda^\dagger)\} - b_K\frac{i}{4}(\lambda\eta^\dagger-\eta\lambda^\dagger)(\lambda\lambda^\dagger) + \frac{i}{2}\left(\lambda\sigma_a\eta^\dagger - \eta\sigma_a\lambda^\dagger\right)\mathcal{S}_{ab}b_b \tag{A.27}$$

$$\begin{aligned} N_K = {} & b_K\{1+(\lambda\eta^\dagger+\eta\lambda^\dagger) - \frac{1}{2}(\eta\sigma_a\eta^\dagger)(\lambda\sigma_a\lambda^\dagger) + \frac{1}{4}(\eta\eta^\dagger)(\lambda\lambda^\dagger)(\lambda\eta^\dagger+\eta\lambda^\dagger) + \frac{1}{16}(\eta\eta^\dagger)^2(\lambda\lambda^\dagger)^2\} \\ & + b_H\frac{1}{4}(\eta\eta^\dagger)^2 + b_D\frac{i}{2}(\eta\eta^\dagger)(\lambda\eta^\dagger - \eta\lambda^\dagger) \\ & -\{(\eta\sigma_a\eta^\dagger) + \frac{1}{2}(\eta\eta^\dagger)\left(\lambda\sigma_a\eta^\dagger+\eta\sigma_a\lambda^\dagger\right) + \frac{1}{4}(\eta\eta^\dagger)^2(\lambda\sigma_a\lambda^\dagger)\}\mathcal{S}_{ab}b_b \end{aligned} \tag{A.28}$$

$$N_{rest} = b_H\{-\frac{i}{2}(\eta d\eta^\dagger - d\eta\eta^\dagger)\} + b_D\{(\lambda d\eta^\dagger + d\eta\lambda^\dagger)\}$$
$$+ b_K\{-\frac{i}{2}(\lambda d\lambda^\dagger - d\lambda\lambda^\dagger) - \frac{i}{2}(\lambda d\eta^\dagger - d\eta\lambda^\dagger)(\lambda\lambda^\dagger) - \frac{i}{8}(\eta d\eta^\dagger - d\eta\eta^\dagger)(\lambda\lambda^\dagger)^2\}$$
$$+ \{i(\lambda\sigma_a d\eta^\dagger - d\eta\sigma_a\lambda^\dagger) + \frac{i}{2}(\eta d\eta^\dagger - d\eta\eta^\dagger)(\lambda\sigma_a\lambda^\dagger)\}\mathcal{S}_{ab}b_b. \tag{A.29}$$

**MC forms in the AdS basis**

The bosonic part of the MC forms in the AdS basis are given as

$$L_0^{P_0} = dx^0 \cosh\frac{x^1}{R}, \qquad L_0^{P_1} = dx^1, \qquad L_0^{M_{01}} = \frac{dx^0}{R}\sinh\frac{x^1}{R}, \tag{A.30}$$
$$L_0^{J_1} = d\theta^1 \cos\theta^2, \qquad L_0^{J_2} = d\theta^2, \qquad L_0^{J_3} = -d\theta^1 \sin\theta^2. \tag{A.31}$$

The $L_1^{P_0}, L_1^{P_1}, L_1^{M_{01}}, L_1^{J_b}$ in (5.7) are including the fermionic contributions and are obtained as

$$L_1^{P_0} = R(L_1^H + L_1^K), \qquad L_1^{P_1} = RL_1^D, \qquad L_1^{M_{01}} = L_1^K - L_1^H, \qquad L_1^{J_b}. \tag{A.32}$$

Here $L_1^H, L_1^K, L_1^D, L_1^{J_b}$ are obtained from $L^H, L^K, L^D, L^{J_b}$ in the conformal basis (A.11)-(A.14) in which $L_H^0, L_K^0, L_D^0, L_{J_b}^0$ are replaced by

$$L_H^0 \to \frac{L_0^{P_0}}{2R} - \frac{L_0^{M_{01}}}{2}, \qquad L_K^0 \to \frac{L_0^{P_0}}{2R} + \frac{L_0^{M_{01}}}{2}, \qquad L_D^0 \to \frac{L_0^{P_1}}{R}, \qquad L_{J_b}^0 \to L_0^{J_b}, \tag{A.33}$$

where the bosonic MC one forms in the AdS base are given in (A.30) and (A.31).

## B. PSU(1,1|2) transformations

The bosonic transformations of $PSU(1,1|2)$ are given in (2.7)-(2.10). Supersymmetric and superconformal transformations of the goldstone fields can be calculated in the same way, obtaining however, complicated expressions. It is convenient to give them here for further references.

- Ordinary supersymmetry:

$$\delta_Q t = \frac{i}{2}e^{z/2}\left[-\eta^\dagger + i\frac{1}{2}\omega\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\right]\epsilon_Q$$
$$\delta_Q z = \frac{1}{2}\omega^2 e^{-z/2}\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\epsilon_Q$$
$$\delta_Q \omega = -\frac{i}{2}e^{-z/2}\omega^2\left[\eta^\dagger + i\frac{1}{2}\omega\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\right]\epsilon_Q - \frac{\omega\lambda^\dagger}{2\Delta}e^{-z/2}\epsilon_Q \tag{B.1}$$
$$\delta_{Q_i}\eta_k = e^{-z/2}\left(\delta_{ik} + \frac{i}{2}\omega\left(\eta\eta^\dagger\right)\left(-\delta_{ik} + \frac{\lambda_i^\dagger\eta_k}{\Delta}\right)\right)\epsilon_{Q_i}$$
$$\delta_{Q_i}\eta_k^\dagger = i\omega e^{-z/2}\left(\eta^\dagger - \frac{\lambda_i^\dagger}{\Delta}\left(\eta\eta^\dagger\right)\right)\eta_k^\dagger\epsilon_{Q_i}$$

$$\delta_{Q_i}\lambda_k = i\omega e^{-z/2}\left(\delta_{ik} - \frac{\lambda_i^\dagger \eta_k}{2\Delta}\right)\epsilon_{Q_i}$$

$$\delta_{Q_i}\lambda_k^\dagger = i\omega e^{-z/2}\frac{\lambda_i^\dagger \eta_k^\dagger}{2\Delta}\epsilon_{Q_i} \tag{B.2}$$

$$\delta_Q \theta^b = -i\omega e^{-z/2}\left(i\sigma_a\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(i\eta\sigma_a\eta^\dagger\right)\right)\mathcal{P}^{ab}\epsilon_{Q_i} \tag{B.3}$$

Where

$$\Delta = 1 + \frac{\eta\lambda^\dagger + \lambda\eta^\dagger}{2}. \tag{B.4}$$

The transformations under $Q^\dagger$ are obtained by taking conjugations. For example from (B.2)

$$\delta_{Q_i^\dagger}\eta_k^\dagger = e^{-z/2}\left(\delta_{ik} + \tfrac{i}{2}\omega\left(\eta\eta^\dagger\right)\left(\delta_{ik} + \tfrac{\lambda_i\eta_k^\dagger}{\Delta}\right)\right)\epsilon_{Q_i^\dagger}. \tag{B.5}$$

- Superconformal transformations

$$\delta_S t = \frac{i}{2}e^{z/2}\left[it\eta^\dagger + \frac{1}{2}\left(e^z + t\omega\right)\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\right]\epsilon_S$$

$$\delta_S z = -\frac{i}{2}\omega e^{-z/2}\left(e^z + t\omega\right)\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\epsilon_S - e^{z/2}\eta^\dagger\epsilon_S$$

$$\begin{aligned}\delta_S\omega = {}& \frac{i}{2}e^{-z/2}\omega^2\left[it\eta^\dagger - \frac{1}{2}\left(e^z + t\omega\right)\left(\eta\eta^\dagger\right)\left(\eta^\dagger - \frac{\lambda^\dagger}{2\Delta}\left(\eta\eta^\dagger\right)\right)\right]\epsilon_S - e^{z/2}\eta^\dagger\omega\epsilon_S \\ & + i\frac{\lambda^\dagger}{2\Delta}\left(e^{z/2} + t\omega e^{-z/2}\right)\epsilon_S\end{aligned} \tag{B.6}$$

$$\delta_{S_i}\eta_k = \left(-ite^{-z/2}\delta_{ik} + \frac{1}{2}\left(e^{z/2} + t\omega e^{-z/2}\right)\left(\eta\eta^\dagger\right)\left(-\delta_{ik} + \frac{\lambda_i^\dagger\eta_k}{\Delta}\right)\right)\epsilon_{S_i} \tag{B.7}$$

$$\delta_{S_i}\eta_k^\dagger = \left(e^{z/2} + t\omega e^{-z/2}\right)\left(\eta\eta^\dagger\right)\left(\eta_i^\dagger - \frac{\lambda_i^\dagger}{\Delta}\left(\eta\eta^\dagger\right)\right)\eta_k^\dagger\epsilon_{S_i}$$

$$\delta_{S_i}\lambda_k = \left(e^{z/2} + t\omega e^{-z/2}\right)\left(\delta_{ik} - \frac{\lambda_i^\dagger\eta_k}{2\Delta}\right)\epsilon_{S_i}$$

$$\delta_{S_i}\lambda_k^\dagger = \left(e^{z/2} + t\omega e^{-z/2}\right)\frac{\lambda_i^\dagger\eta_k^\dagger}{2\Delta}\epsilon_{S_i}$$

$$\delta_S\theta^b = -\left(e^{z/2} + t\omega e^{-z/2}\right)\left(i\sigma_a\eta^\dagger - \frac{\lambda^+}{2\Delta}\left(i\eta\sigma_a\eta^\dagger\right)\right)\mathcal{P}^{ab}\epsilon_S \tag{B.8}$$

## C. Diffeomorphism in terms of the gauge symmetries

It is shown here that the diffeomorphism of the action (3.2) is equivalent to a suitable combination of the $T$-gauge (4.11), U(1) (4.12) and kappa (4.14) transformations for the BPS case $b_H b_K = b_D^2 + b_a b_a$, (4.13).



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### C.1 Trivial Symmetry

The Euler derivatives $(\mathcal{L})_M$ are defined as

$$\delta\mathcal{L} = (\mathcal{L})_M \delta Z^M + \text{surface term}, \tag{C.1}$$

any action is invariant under a transformation of the form

$$\delta Z^M = (\mathcal{L})_N A^{NM}, \qquad A^{MN} = -(-)^{MN} A^{NM}, \tag{C.2}$$

that is, $A^{MN}$ is graded anti-symmetric. $(-)^{MN} = -1$ when both $M$ and $N$ are odd indices and $(-)^{MN} = +1$ otherwise. It is a trivial symmetry and does not lead to a Noether charge. Now the Lagrangian is (3.2).

$$\mathcal{L} = b_A L^A. \tag{C.3}$$

In this appendix the pullback on $L^A$ is tacitly understood. The Euler derivative $(\mathcal{L})_M$ is

$$(\mathcal{L})_M = b_A f^A_{BC} (\dot{Z}^N L_N{}^C)(L_M{}^B)(-1)^{M(M+B)}. \tag{C.4}$$

In the present formulation we use all group coordinates $Z^M$ the $L'_M{}^B \equiv (L_M{}^B)(-1)^{M(M+B)}$ has the inverse $L'_B{}^M$. It is convenient to define

$$[\mathcal{L}]_B = (\mathcal{L})_M L'_B{}^M = b_A f^A_{BC} (\dot{Z}^N L_N{}^C). \tag{C.5}$$

Using it (C.1) is written as

$$\delta\mathcal{L} = [\mathcal{L}]_A [\delta Z^A] + \text{surface term}. \tag{C.6}$$

Then a transformation is trivial if $[\delta Z^A]$ is written as a (graded) antisymmetric combination of the equations of motion (C.5),

$$[\delta Z^A] = [\mathcal{L}]_B \tilde{A}^{BA}, \qquad \tilde{A}^{AB} = -(-)^{AB} \tilde{A}^{BA}. \tag{C.7}$$

### C.2 Geometrical diffeomorphism

For the geometrical diffeomorphism

$$\delta_{diff} Z^M = \varepsilon \dot{Z}^M, \quad \to \quad \left[\delta_{diff} Z^A\right] = \varepsilon \dot{Z}^M \; L_M{}^A = \varepsilon L^A \tag{C.8}$$

We will show the geometrical diffeomorphism is not independent of the gauge transformations but equivalent to a combination of the gauge transformations. More precisely they differ by a trivial transformation discussed above.

The gauge transformations of (3.3-6) is

$$\begin{aligned}
&[\delta_{gauge} t] = \epsilon(\tau), \quad [\delta_{gauge} z] = -2\frac{b_D}{b_K}\epsilon(\tau), \quad [\delta_{gauge} w] = \frac{b_H}{b_K}\epsilon(\tau), \\
&[\delta_{gauge} \theta^a] = b_{J_a} \alpha(\tau),
\end{aligned}$$

$$[\delta_{gauge}\eta] = \kappa_\eta(\tau)s(\theta), \qquad [\delta_{gauge}\lambda] = \kappa_\eta(\tau)s(\theta)(\frac{ib_D}{b_K} + \frac{b_a\sigma_a}{b_K}). \tag{C.9}$$

Let $\Delta$ is difference of "$\delta_{diff}$" and "$\delta_{gauge}$",

$$\begin{aligned}
[\Delta t] &= = \varepsilon L^H - \epsilon(\tau), \quad [\Delta z] = \varepsilon L^D + 2\frac{b_D}{b_K}\epsilon(\tau), \\
[\Delta w] &= \varepsilon L^K - \frac{b_H}{b_K}\epsilon(\tau), \quad [\Delta\theta^a] = \varepsilon L^{J_a} - b_{J_a}\alpha(\tau), \\
[\Delta\eta] &= \varepsilon L^Q - \kappa_\eta(\tau)s(\theta), \qquad [\Delta\lambda] = \varepsilon L^S - \kappa_\eta(\tau)s(\theta)(\frac{ib_D}{b_K} + \frac{b_a\sigma_a}{b_K}),
\end{aligned} \tag{C.10}$$

We choose the gauge parameter functions $\epsilon, \alpha, \kappa$ as

$$\epsilon(\tau) = \varepsilon L^H, \qquad \alpha(\tau) = \varepsilon\frac{(b_b L^b)^*}{b_c^2}, \qquad \kappa_\eta(\tau)s(\theta) = \varepsilon L^Q \tag{C.11}$$

so that, using Euler derivatives in (C.5),

$$\begin{aligned}
[\Delta t] &= 0, \\
[\Delta z] &= \frac{\varepsilon}{b_K}\left(b_K L^D + 2b_D L^H\right) = -\frac{\varepsilon}{b_K}[\mathcal{L}]_K, \\
[\Delta w] &= \frac{\varepsilon}{b_K}\left(b_K L^K - b_H L^H\right) = \frac{\varepsilon}{b_K}[\mathcal{L}]_D, \\
[\Delta\theta^a] &= \frac{\varepsilon}{b_c^2}\epsilon_{abc}b_c(\epsilon_{bde}b_d L^e) = -\frac{\varepsilon}{b_c^2}\epsilon_{abc}b_c\,[\mathcal{L}]_b, \\
[\Delta\eta] &= 0, \\
[\Delta\lambda] &= \frac{\varepsilon}{b_K}\left(b_K L^S - L^Q(ib_D + b_a\sigma_a)\right) = -i\frac{\varepsilon}{b_K}\,[\mathcal{L}]_S.
\end{aligned} \tag{C.12}$$

We also have for the conjugate coordinates

$$\left[\Delta\eta^\dagger\right] = 0, \qquad \left[\Delta\lambda^\dagger\right] = i\frac{\varepsilon}{b_K}\,[\mathcal{L}]_{S^\dagger}. \tag{C.13}$$

From (2.1) remembering that the coordinate for $S$ is $\lambda^\dagger$ while that of $S^\dagger$ is $-\lambda$ they are written in the matrix form

$$\begin{pmatrix} [\Delta t] \\ [\Delta z] \\ [\Delta w] \\ [\Delta\theta_a] \\ [\Delta\eta^\dagger] \\ -[\Delta\eta] \\ [\Delta\lambda^\dagger] \\ -[\Delta\lambda] \end{pmatrix}^T = \begin{pmatrix} [\mathcal{L}]_H \\ [\mathcal{L}]_D \\ [\mathcal{L}]_K \\ [\mathcal{L}]_{J_b} \\ [\mathcal{L}]_Q \\ [\mathcal{L}]_{Q^\dagger} \\ [\mathcal{L}]_S \\ [\mathcal{L}]_{S^\dagger} \end{pmatrix}^T \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{\varepsilon}{b_K} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{\varepsilon}{b_K} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\frac{\varepsilon}{b_d^2}\epsilon_{abc}b_c & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i\frac{\varepsilon}{b_K} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i\frac{\varepsilon}{b_K} & \cdot \end{pmatrix}. \tag{C.14}$$

The matrix appearing here is graded anti-symmetric and the transformation $\Delta$ is shown to be trivial.

In the non-BPS case, $b_H b_K - b_D^2 \neq b_{J_a}^2$, there is no kappa symmetry and $\kappa_\eta$ is taken to be zero in (C.10) and ( C.11). In this case

$$[\Delta\eta] = \varepsilon L^Q = -i\varepsilon \frac{[\mathcal{L}]_Q\, b_K - [\mathcal{L}]_S\,(ib_D - b_a\sigma_a)}{b_H b_K - b_D^2 - b_{J_a}^2},$$
$$[\Delta\lambda] = \varepsilon L^S = -i\varepsilon \frac{[\mathcal{L}]_S\, b_H + [\mathcal{L}]_S\,(ib_D + b_a\sigma_a)}{b_H b_K - b_D^2 - b_{J_a}^2}. \tag{C.15}$$

They are also graded anti-symmetric combinations of the equations of motion and the difference of the diffeomorphism and the $H$ and U(1) transformations is a trivial transformation.

## D. Conformal mechanics invariant under $OSP(2|2)$

In this appendix we explicitly derive the kappa transformation of the $OSP(2|2)$ case in an arbitrary configuration. Furthermore kappa invariant and quasi invariant variables are constructed and the lagrangian is written in terms of them. To show the relation with the former case a dictionary is given.

The OSP(2|2) is a subalgebra of SU( 1, 1| 2) whose generators are

$$H, \quad K, \quad D, \quad B = -2J_2 \tag{D.1}$$

and

$$\mathbf{Q}_i = \frac{1}{\sqrt{2}}(Q^i + Q^\dagger{}_i), \qquad \mathbf{S}_i = \frac{i}{\sqrt{2}}(S^i - S^\dagger{}_i). \tag{D.2}$$

They satisfy $OSP(2|2)$ algebra:

$$[H, D] = iH \qquad [K, D] = -iK \qquad [H, K] = 2iD \tag{D.3}$$
$$[\mathbf{Q}_i, \mathbf{Q}_j]_+ = \delta_{ij} H \qquad [\mathbf{S}_i, \mathbf{S}_j]_+ = \delta_{ij} K \qquad [\mathbf{Q}_i, \mathbf{S}_j]_+ = \delta_{ij} D + \frac{1}{2}\varepsilon_{ij} B \tag{D.4}$$
$$[D, \mathbf{Q}_i] = -\frac{i}{2}\mathbf{Q}_i \qquad [D, \mathbf{S}_i] = \frac{i}{2}\mathbf{S}_i \qquad [K, \mathbf{Q}_i] = -i\mathbf{S}_i \tag{D.5}$$
$$[H, \mathbf{S}_i] = i\mathbf{Q}_i \qquad [B, \mathbf{Q}_i] = -i\varepsilon_{ij}\mathbf{Q}_j \qquad [B, \mathbf{S}_i] = -i\varepsilon_{ij}\mathbf{S}_j \tag{D.6}$$

The group element is parametrized as

$$g = e^{-itH} e^{izD} e^{i\omega K} e^{i\tilde{\eta} Q} e^{i\tilde{\lambda} S} e^{i\alpha B} \tag{D.7}$$

All formulas of OSP(2|2) must be given from those of the SU(1,1|2) by the following replacements

$$\eta^i \to -\frac{\tilde{\eta}^i}{\sqrt{2}}, \qquad \eta_i^\dagger \to \frac{\tilde{\eta}^i}{\sqrt{2}}, \qquad \tilde{\eta} = \frac{1}{\sqrt{2}}(\eta_i^\dagger - \eta^i)$$
$$\lambda^i \to \frac{i}{\sqrt{2}}\tilde{\lambda}^i, \qquad \lambda_i^\dagger \to \frac{i}{\sqrt{2}}\tilde{\lambda}^i, \qquad \tilde{\lambda} = \frac{-i}{\sqrt{2}}(\lambda_i^\dagger + \lambda^i)$$
$$\theta^2 = -2\alpha, \quad \mathit{and} \quad \theta^1 = \theta^3 = 0. \tag{D.8}$$

The components of the left invariant Maurer-Cartan form are:

$$
\begin{aligned}
L^H &= -e^{-z}dt + \frac{i}{2}(\eta d\eta) \\
L^K &= d\omega\left(1 + i(\lambda\eta) + \frac{1}{8}(\lambda\epsilon\lambda)(\eta\epsilon\eta)\right) - \omega dz\left(1 + i(\lambda\eta) + \frac{1}{8}(\lambda\epsilon\lambda)(\eta\epsilon\eta)\right) \\
&\quad -\omega^2 e^{-z}dt\left(1 + i(\lambda\eta) + \frac{1}{8}(\lambda\epsilon\lambda)(\eta\epsilon\eta)\right) + \frac{i}{2}(\lambda d\lambda) \\
L^D &= dz\left(1 + \frac{i}{2}(\lambda\eta)\right) + 2\omega e^{-z}dt\left(1 + \frac{i}{2}(\lambda\eta)\right) + i(\lambda d\eta) \\
L^B &= d\alpha + \frac{i}{4}\, d\omega\,(\eta\epsilon\eta) - \frac{i}{4}\, dz\,((\lambda\epsilon\eta) + \omega(\eta\epsilon\eta)) \\
&\quad -e^{-z}dt\frac{i}{4}\left((\lambda\epsilon\lambda) + 2\omega(\lambda\epsilon\eta) + \omega^2(\eta\epsilon\eta)\right) \\
&\quad -\frac{i}{2}(\lambda\epsilon d\eta) - \frac{1}{8}(\lambda\epsilon\lambda)(\eta d\eta) \\
L^Q &= (\cos\alpha + \epsilon\ \sin\alpha)\left[d\eta + \frac{1}{2}\eta\ dz + e^{-z}dt(\lambda + \omega\eta) - \frac{i}{2}\ \lambda(\eta d\eta)\right], \\
L^S &= (\cos\alpha + \epsilon\sin\alpha)\left[d\lambda + (\eta - \frac{i}{4}\epsilon\lambda(\eta\epsilon\eta))\ d\omega - dz\left(\frac{1}{2}\lambda + \eta\omega + \frac{i}{8}\epsilon\eta(\lambda\epsilon\lambda) - \frac{i}{4}\epsilon\lambda\omega(\eta\epsilon\eta)\right)\right. \\
&\quad \left. -e^{-z}dt\left(\omega\lambda + \omega^2\eta + \frac{i}{4}\omega\epsilon\eta(\lambda\epsilon\lambda) - \frac{i}{4}\omega^2\epsilon\lambda(\eta\epsilon\eta)\right) - \frac{i}{4}\ \epsilon d\eta(\lambda\epsilon\lambda)\right],
\end{aligned} \tag{D.9}
$$

where $\epsilon = i\sigma_2$.

Now the action is:

$$
\mathcal{S} = \int \mathcal{L}d\tau = \int \left(b_H L^H + b_K L^K + b_D L^D + b_B L^B\right)^* \tag{D.10}
$$

Under $\kappa$ variations satisfying

$$
[\delta t] = [\delta z] = [\delta\alpha] = [\delta\omega] = 0 \tag{D.11}
$$

the LI one forms transform as:

$$
\delta L^H = -iL^Q\,[\delta\eta] \qquad \delta L^K = -iL^S\,[\delta\lambda] \qquad \delta L^D = -i\left(L^Q\,[\delta\lambda] + L^S\,[\delta\eta]\right) \tag{D.12}
$$

$$
\delta L^B = -\frac{i}{2}\left(L^Q\epsilon\,[\delta\lambda] - L^S\epsilon\,[\delta\eta]\right). \tag{D.13}
$$

The condition for the lagrangian $(D.10)$ to be kappa invariant is given by:

$$
[\delta\eta] = -\frac{1}{b_H}\left(b_D + \frac{1}{2}b_B\epsilon\right)[\delta\lambda] \qquad [\delta\lambda] = -\frac{1}{b_K}\left(b_D - \frac{1}{2}b_B\epsilon\right)[\delta\eta] \tag{D.14}
$$

which in turn implies:

$$
b_K b_H = b_D^2 + \frac{1}{4}b_B^2. \tag{D.15}
$$

When this is verified it is kappa symmetric else it describes non-BPS paricle.

The kappa transformations of the BPS particle are found as follows. Due to the former condition, (D.11), we can find the explicit form of the kappa variations for the bosonic fields in term of the fermionic ones:

$$
\begin{aligned}
\delta_\kappa t &= \frac{i}{2} e^z \eta \delta_\kappa \eta \\
\delta_\kappa z &= -\left(1 - \frac{i}{2}(\lambda\eta)\right) i\lambda\delta_\kappa\eta - \omega i \eta \delta_\kappa \eta \\
\delta_\kappa \omega &= -\left(1 - i(\lambda\eta)\right) \frac{i}{2}\lambda\delta_\kappa\lambda - \left(1 - \frac{i}{2}(\lambda\eta)\right) i\omega\lambda\delta_\kappa\eta - \frac{i}{2}\omega^2\eta\delta_\kappa\eta \\
\delta_\kappa \alpha &= \frac{i}{2}(\lambda\epsilon\delta_\kappa\eta) - \frac{1}{8}(\eta\epsilon\eta)(\lambda\delta_\kappa\lambda) + \frac{1}{4}(\lambda\epsilon\eta)(\lambda\delta_\kappa\eta).
\end{aligned}
\tag{D.16}
$$

Introducing kappa parameters:

$$
[\delta\eta] = (\cos\alpha + \epsilon \sin\alpha)\,\kappa_\eta \qquad [\delta\lambda] = (\cos\alpha + \epsilon\sin\alpha)\,\kappa_\lambda, \tag{D.17}
$$

we get

$$
\delta_\kappa \eta = \kappa_\eta + \frac{i}{2}\eta\,(\lambda\kappa_\eta) \qquad \delta_\kappa\lambda = \kappa_\lambda + \frac{i}{2}\eta\,(\lambda\kappa_\lambda)\,. \tag{D.18}
$$

(D.14) is solved for $\kappa_\lambda$ as

$$
\kappa_\lambda = -\frac{1}{b_K}(b_D - \frac{1}{2}b_B\epsilon)\kappa_\eta. \tag{D.19}
$$

We can introduce the kappa invariant variables; fermionic coordinates:

$$
\Psi = (\lambda + \frac{1}{b_K}(b_D - \frac{b_B}{2}\epsilon)\eta) + \frac{ib_B}{4b_K}(\lambda\eta)\epsilon\eta. \tag{D.20}
$$

and the bosonic coordinate:

$$
q = \sqrt{2} e^{\frac{z}{2}} \left(\frac{N_K}{b_K}\right)^{\frac{1}{2}}. \tag{D.21}
$$

Using the kappa condition $(D.15)$ the lagrangian is expressed in terms of the kappa invariant variables:

$$
\mathcal{L} = b_K \frac{\dot{q}^2}{2\dot{t}} - \frac{2\dot{t}}{b_K q^2}\left(\frac{b_B^2}{4} + \frac{ib_B b_K}{4}(\Psi\epsilon\Psi)\right) + b_K\frac{i}{2}\Psi\dot{\Psi} + (\text{surface term}). \tag{D.22}
$$

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